

λ^* -CLOSED SETS AND NEW SEPARATION AXIOMS IN ALEXANDROFF SPACES

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ABSTRACT. Here we have studied the ideas of g^* -closed sets, $g \wedge_\tau$ -sets and λ^* -closed sets and investigate some of their properties in the spaces of A. D. Alexandroff [1]. We have also studied few separation axioms like $T_{\frac{\omega}{4}}$, $T_{\frac{3\omega}{8}}$, T_ω in Alexandroff spaces and have introduced a new separation axiom namely $T_{\frac{5\omega}{8}}$ axiom in this space.

1. Introduction

Topological spaces have been generalised in many ways. A. D. Alexandroff (1940) weakened the union requirements where only countable union of open sets were taken to be open.

The idea of generalised closed sets in a topological space was given by Levine [8]. Later many works on generalised closed sets have been done [2], [3], [5], [6] etc. In 2003, P. Das et al obtained a generalisation of closed sets in Alexandroff spaces which was called g^* -closed sets. They investigated various properties on g^* -closed sets and also introduced a new separation axiom namely T_w axiom in Alexandroff spaces exactly in the same way as Levine [8] defined $T_{\frac{1}{2}}$ -spaces in topological spaces. Recently M. S. Sarsak [10] studied the same in a generalised topological space where a generalised topology μ on a nonempty set X is a collection of subsets of X such that $\emptyset \in \mu$ and μ is closed under arbitrary unions. Members of μ are called μ -open sets. He also introduced some new separation axioms namely μ - $T_{\frac{1}{4}}$, μ - $T_{\frac{3}{8}}$ and μ - $T_{\frac{1}{2}}$ axioms and studied their properties and relations between the axioms.

Here we have studied the idea of generalised closed sets namely the g^* -closed sets in Alexandroff spaces. We have investigated few more properties of g^* -closed sets which were not studied in [5]. Also we have obtained few separation axioms like $T_{\frac{\omega}{4}}$, $T_{\frac{3\omega}{8}}$, T_ω in Alexandroff spaces in the same way as that of Sarsak [10] and investigate how far several results as valid in [10] are affected in Alexandroff spaces. Also we have introduced a new separation

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axiom namely $T_{\frac{5\omega}{8}}$ axiom in a Alexandroff space which was not studied before. We have found out that $T_{\frac{5\omega}{8}}$ -space can be placed in between $T_{\frac{3\omega}{8}}$ and T_ω spaces.

2. Preliminaries

Definition 2.1 [1]: A non empty set X is called a σ -space or simply a space if in it is chosen a system of subsets \mathcal{F} satisfying the following axioms:

- (1) The intersection of a countable number of sets in \mathcal{F} is a set in \mathcal{F} .
- (2) The union of a finite number of sets in \mathcal{F} is a set in \mathcal{F} .
- (3) The void set is a set in \mathcal{F} .
- (4) The whole set X is a set in \mathcal{F} .

Sets of \mathcal{F} are called closed sets. Their complementary sets are called open sets. The collection of all such open sets will sometimes be denoted by τ and the space by (X, τ) . When there is no confusion, the space (X, τ) will simply be denoted by X .

Note that a topological space is a space but in general τ is not a topology as can be easily seen by taking $X = R$ and τ as the collection of all F_σ -sets in R . Several examples of spaces are seen in [4], [5], [7]. The definition of closure of a set and interior of a set in a space are similar as in the case of a topological space. Note that closure of a set in a space may not be closed in general. Also interior of a set in a space may not be open.

Throughout the paper X stands for a space and sets are always subsets of X unless otherwise stated. The letters R and Q stand respectively for the set of real numbers and the set of rational numbers.

Definition 2.2 [6] : Two sets A, B in X are said to be weakly separated if there are two open sets U, V such that $A \subset U, B \subset V$ and $A \cap V = B \cap U = \emptyset$.

Definition 2.3 [1] : A space (X, τ) is called T_0 if for any pair of distinct points of X , there is an open set containing one of the points but not the other.

Observe that a space (X, τ) is T_0 if and only if for any pair of distinct points $x, y \in X$, there is a set A containing one of the points such that A is either open or closed.

Clearly it can be easily checked that if a space (X, τ) is T_0 then for every pair of distinct points $p, q \in X$, either $p \notin \overline{\{q\}}$ or $q \notin \overline{\{p\}}$.

Definition 2.4 [1] : (X, τ) is said to be T_1 -space if for any pair of distinct points $x, y \in X$, there exist open sets U, V such that $x \in U, y \notin U, y \in V, x \notin V$.

Definition 2.5 [5] : A point $x \in X$ is said to be a limit point of A in a space (X, τ) if for any open set U containing x , $U \cap (A - \{x\}) \neq \emptyset$. The set of all limit points of A is called derived set of A and is denoted by A' .

Definition 2.6 [9] : A space (X, τ) is said to be a R_0 -space if every open set contains the closure of each of its singleton.

Definition 2.7 [1] : A space (X, τ) is said to be bicomact if every open cover of it has a finite subcover.

Theorem 2.8 : Let A be a subset in a space (X, τ) , then $X - \overline{X - A} = \text{Int}(A)$.

The proof is straightforward and so is omitted.

3. g^* -closed sets, $g \wedge_\tau$ -sets and $g \vee_\tau$ -sets in a space

Definition 3.1 [8]: A set A in a topological space is said to be a generalised closed (g -closed for short) if and only if $\overline{A} \subset U$ whenever $A \subset U$ and U is open.

Definition 3.2 : (cf. Definition 9 [5]): A set A in a space is said to be a g^* -closed set if and only if there is a closed set F containing A such that $F \subset U$ whenever $A \subset U$ and U is open.

Clearly every closed set is g^* -closed but the converse may not be true as shown in the Example 1 [5].

We now state few important theorems given in [5].

Theorem 3.3 [5]: A set A is g^* -closed if and only if there is a closed set F containing A such that $F \subset \ker(A)$.

Theorem 3.4 [5] : For each $x \in X$, $\{x\}$ is either closed or $X - \{x\}$ is g^* -closed.

Theorem 3.5 [5] : A set A is g^* -closed if and only if there is a closed set F containing A such that $F - A$ does not contain any non-void closed set.

Definition 3.6 [5]: A set A is called g^* -open set if $X - A$ is g^* -closed set.

Theorem 3.7 [5] : A set A is g^* -open set if and only if there is an open set V contained in A such that $F \subset V$ whenever F is closed and $F \subset A$.

Definition 3.8 (cf.[5]) : Let $A \subset X$. The kernel of A is the set $\cap\{U : U \in \tau, A \subset U\}$ and denoted by A_τ^\wedge .

Definition 3.9 (cf. [10]) : Let $A \subset X$. Then we define $A_\tau^\vee = \cup\{F : X - F \in \tau, F \subset A\}$.

Definition 3.10 (cf. [10]): A set A is called a \wedge_τ -set if $A = A_\tau^\wedge$ or, equivalently, A is the intersection of all open sets containing A .

Definition 3.11 (cf. [10]): A set A is called a \vee_τ -set if $A = A_\tau^\vee$ or, equivalently, $X - A$ is a \wedge_τ -set (i.e. A is the union of all closed sets contained in A). Clearly X and \emptyset are \vee_τ -sets as well as \wedge_τ -sets.

Clearly an open set is g^* -open set. But the converse may not be true as shown in the following example.

Example 3.12: Let $X = R - Q$, $\tau = \{X, \emptyset, G_i\}$, Where G_i be the all countable subsets in X . Thus (X, τ) is a space but not a topological space. Let B be the set of all positive irrational numbers. Then B is g^* -closed, since X is the only open and closed set containing B . Then $X - B$ is g^* -open but is not an open set.

Theorem 3.13 : Intersection of two g^* -open sets is g^* -open.

Proof. Let A, B be two g^* -open sets. There exist open sets U, V contained in A, B respectively such that $F_1 \subset U$ and $F_2 \subset V$ whenever $F_1 \subset A$ and $F_2 \subset B$; F_1 and F_2 are closed sets. Clearly $U \cap V$ is open and $U \cap V \subset A \cap B$. Let F be a closed set such that $F \subset A \cap B$. Then $F \subset A$ and $F \subset B$. So $F \subset U$ and $F \subset V$. Therefore $F \subset U \cap V$ which implies $A \cap B$ is g^* -open. \square

Theorem 3.14 : A set A is g^* -open if and only if there is an open set $V \subset A$ such that $A_\tau^\vee \subset V$.

Proof. Let A be a g^* -open set. Then by Theorem 3.7, there exists an open set $V \subset A$ such that $F \subset V$ whenever $F \subset A$ and F is closed. So $\cup\{F : F \subset A, F \text{ is closed}\} \subset V$. This implies that $A_\tau^\vee \subset V$.

Conversely, let there be an open set $V \subset A$ such that $A_\tau^\vee \subset V$. This implies that $A_\tau^\vee = \cup\{F : F \subset A, F \text{ is closed}\} \subset V$. So there is an open set V such that $F \subset V$ whenever $F \subset A, F$ is closed. Hence A is g^* -open by Theorem 3.7. \square

Definition 3.15 (cf. Definition 2.11[10]) : A set A is called a generalised \wedge_τ -set denoted as $g\wedge_\tau$ -set if $A_\tau^\wedge \subset F$ whenever $F \supset A$ and F is closed. A set A is called a generalised \vee_τ -set denoted as $g\vee_\tau$ -set if $X - A$ is $g\wedge_\tau$ -set.

Theorem 3.16: A set A is a $g\vee_\tau$ -set if and only if $A_\tau^\vee \supset U$ whenever $A \supset U$ and U is open. The proof is simple and so is omitted.

Clearly if A is a \wedge_τ -set then A is $g\wedge_\tau$ -set. But the converse is not always true as shown in the following example.

Example 3.17: Let $X = R$, G_i be the all countable subsets of irrationals in R and $\tau = \{X, \emptyset, G_i\}$. Therefore (X, τ) is a space but not a topological space. Suppose $A = R - Q$. Therefore $A_\tau^\wedge = X$ and $\overline{A} = X$. Therefore A is $g\wedge_\tau$ -set, since the only closed set containing A is X . But $A = R - Q \neq X = A_\tau^\wedge$. This implies that A is not \wedge_τ -set.

Lemma 3.18 : Let A, B be subsets of X . Then the following hold:

- (1) $\emptyset_\tau^\wedge = \emptyset, \quad \emptyset_\tau^\vee = \emptyset, \quad X_\tau^\wedge = X, \quad X_\tau^\vee = X.$
- (2) $A \subset A_\tau^\wedge, \quad A_\tau^\vee \subset A.$
- (3) $(A_\tau^\wedge)^\wedge = A_\tau^\wedge, \quad (A_\tau^\vee)^\vee = A_\tau^\vee.$
- (4) $A \subset B \Rightarrow A_\tau^\wedge \subset B_\tau^\wedge.$
- (5) $A \subset B \Rightarrow A_\tau^\vee \subset B_\tau^\vee.$

$$(6) (X - A)_\tau^\wedge = X - A_\tau^\vee, \quad (X - A)_\tau^\vee = X - A_\tau^\wedge.$$

Theorem 3.19 : Arbitrary union of \vee_τ -sets is a \vee_τ -set.

Proof. Suppose that A_i 's are \vee_τ -sets, $i \in I$ where i is an index set and $A = \cup\{A_i : i \in I\}$. So $A_i \subset A$ for each i . Therefore $A_i^\vee \subset A_\tau^\vee$ for all $i \in I$ and so $\cup\{A_i^\vee : i \in I\} \subset A_\tau^\vee$. So, $\cup\{A_i : i \in I\} = \cup\{A_i^\vee : i \in I\} \subset A_\tau^\vee \subset A$ by Lemma 3.18 (2). Therefore $A = A_\tau^\vee$. \square

Corollary 3.20 : Arbitrary intersection of \wedge_τ -sets is a \wedge_τ -set.

Proof. Let $\{B_i : i \in \Lambda\}$ be any collection of \wedge_τ -sets and let $B = \cap_{i \in \Lambda} B_i$. Then $X - B = X - \cap_{i \in \Lambda} B_i = \cup_{i \in \Lambda} (X - B_i)$. Since each $X - B_i$ is \vee_τ -sets, by Theorem 3.19 $X - B$ is \vee_τ -sets and so B is \wedge_τ -set. \square

Theorem 3.21 : Intersection of two \vee_τ -sets is a \vee_τ -set.

Proof. Let A, B be two \vee_τ -sets. Then $A = A_\tau^\vee, B = B_\tau^\vee$. Now $A \cap B \subset A$ and $A \cap B \subset B$. Therefore $(A \cap B)_\tau^\vee \subset A_\tau^\vee$ and $(A \cap B)_\tau^\vee \subset B_\tau^\vee$. So $(A \cap B)_\tau^\vee \subset A_\tau^\vee \cap B_\tau^\vee = A \cap B$.

Conversely, suppose that $x \in A_\tau^\vee \cap B_\tau^\vee$. Since $x \in A_\tau^\vee$ and $x \in B_\tau^\vee$, there exist closed sets F, P such that $x \in F \subset A, x \in P \subset B$. Therefore $x \in F \cap P \subset A \cap B$. This implies that $x \in (A \cap B)_\tau^\vee$. Therefore $A_\tau^\vee \cap B_\tau^\vee \subset (A \cap B)_\tau^\vee$. So $A \cap B = A_\tau^\vee \cap B_\tau^\vee = (A \cap B)_\tau^\vee$. \square

Corollary 3.22 : Let (X, τ) be a space. Then the collection of all \vee_τ -sets forms a topology. The proof follows from above two Theorems 3.19 and 3.21 and Lemma 3.18(1).

Remark 3.23 (i) : Note that in view of Theorem 3.3 a set A is g^* -closed if and only if there is a closed set F containing A such that $F \subset A_\tau^\wedge$.

(ii) : Clearly if A is a \wedge_τ -set, then A is g^* -closed if and only if A is closed. In particular $(A_\tau^\wedge)^\wedge = A_\tau^\wedge$ and so A_τ^\wedge is a \wedge_τ -set. Therefore A_τ^\wedge is g^* -closed if and only if A_τ^\wedge is closed.

Theorem 3.24 : If A_τ^\wedge is g^* -closed, then A is g^* -closed.

Proof. Let $A \subset X$ and A_τ^\wedge be g^* -closed. Then by Remark 3.23(i), there exists a closed set F containing A_τ^\wedge i.e. $F \supset A_\tau^\wedge \supset A$ such that $F \subset (A_\tau^\wedge)^\wedge = A_\tau^\wedge$, by Lemma 3.18 (3). So by Remark 3.23(i), A is g^* -closed. \square

Corollary 3.25 : If A_τ^\vee is g^* -open, A is g^* -open.

Proof. If A_τ^\vee is g^* -open then $X - A_\tau^\vee$ is g^* -closed i.e. $(X - A)_\tau^\wedge$ is g^* -closed. So by above Theorem 3.24, $X - A$ is g^* -closed and hence A is g^* -open. \square

But the converse of the above theorem and corollary may not be true in a space as shown in the following example.

Example 3.26 : Suppose that $X = R - Q$, G_i 's are the countable subsets of X , each contains $\sqrt{2}$, and $\tau = \{X, \emptyset, G_i, A_i\}$ where A_i 's are the cocountable subsets of X each contains $\sqrt{2}$. Then (X, τ) is a space but not a topological space. Let A be a countable infinite subset of X excluding $\sqrt{2}$. Then A is a closed set and so A is g^* -closed. But $A_\tau^\wedge = \cap\{\{\sqrt{2}\} \cup (X - \{\alpha\}), \alpha \in X - A, \alpha \neq \sqrt{2}\} = \{\sqrt{2}\} \cup A$ is an open set which is not closed, since $\sqrt{2} \in A_\tau^\wedge$. Since A_τ^\wedge is a \wedge_τ -set, by Remark 3.23(ii), A_τ^\wedge is not g^* -closed.

Therefore $X - A$ is g^* -open, but $(X - A)_\tau^\vee = X - A_\tau^\wedge$ is not g^* -open.

Lemma 3.27 : If A, B are two subsets of X , then $(A \cup B)_\tau^\wedge = A_\tau^\wedge \cup B_\tau^\wedge$.

Proof. Let A, B be two subsets of X . Then $A \subset A \cup B$ implies that $A_\tau^\wedge \subset (A \cup B)_\tau^\wedge$ and $B \subset A \cup B$ implies that $B_\tau^\wedge \subset (A \cup B)_\tau^\wedge$. Therefore $(A_\tau^\wedge \cup B_\tau^\wedge) \subset (A \cup B)_\tau^\wedge$. Again, $A_\tau^\wedge = \cap\{U_i : U_i \supset A, U_i \text{ is open}\}$ and $B_\tau^\wedge = \cap\{V_j : V_j \supset B, V_j \text{ is open}\}$. Therefore $A_\tau^\wedge \cup B_\tau^\wedge = \cap\{(U_i \cup V_j) : U_i \supset A, V_j \supset B, U_i, V_j \text{ are open}\} \supset \cap\{G : G \supset A \cup B, G \text{ is open}\} = (A \cup B)_\tau^\wedge$. Therefore $A_\tau^\wedge \cup B_\tau^\wedge = (A \cup B)_\tau^\wedge$. \square

Theorem 3.28 : Union of finite number of \wedge_τ -sets is a \wedge_τ -set.

The proof is straightforward by Lemma 3.27 and so is omitted.

But arbitrary union of \wedge_τ -sets is not always a \wedge_τ -set as can be seen from the example given below.

Example 3.29 : Let $X = R - Q$, G_i be the all countable subsets of X and $\tau = \{X, \emptyset, G_i\}$. Then (X, τ) is a space but not a topological space. Every singleton is an open set, so a \wedge_τ -set. Now the set $A = [1, 2] - Q$ is not open and $A_\tau^\wedge = X \neq A$. So A is not a \wedge_τ -set. But $A = \cup\{\{r\} : r \in A\}$ where $\{r\}$ is \wedge_τ -set.

Theorem 3.30 : Union of two $g\wedge_\tau$ -sets is $g\wedge_\tau$ -set.

Proof. Suppose A, B be two $g\wedge_\tau$ -sets of X , then $A_\tau^\wedge \subset F_1$ whenever F_1 is closed and $A \subset F_1$ and $B_\tau^\wedge \subset F_2$ whenever F_2 is closed and $B \subset F_2$. Let $(A \cup B) \subset F$, F is closed, then $A \subset F$ and $B \subset F$ which imply that $A_\tau^\wedge \subset F$ and $B_\tau^\wedge \subset F$. So $(A_\tau^\wedge \cup B_\tau^\wedge) \subset F$. Therefore by Lemma 3.27, $(A \cup B)_\tau^\wedge = (A_\tau^\wedge \cup B_\tau^\wedge) \subset F$. Hence $A \cup B$ is $g\wedge_\tau$ -set. \square

Corollary 3.31: Intersection of two $g\vee_\tau$ -sets is $g\vee_\tau$ -set.

Proof. Let A, B be two $g\vee_\tau$ -sets. Therefore $(X - A)$ and $(X - B)$ are two $g\wedge_\tau$ -sets. Then by Theorem 3.30, $(X - A) \cup (X - B)$ is a $g\wedge_\tau$ -set i.e. $X - (A \cap B)$ is a $g\wedge_\tau$ -set. Hence $A \cap B$ is a $g\vee_\tau$ -set. \square

But union of two $g\vee_\tau$ -sets is not always $g\vee_\tau$ -set as shown by the example given below.

Example 3.32 : Let $X = R - Q$, G_i 's be the nonempty countable subsets of $X - \{\sqrt{3}\}$ and let $\tau = \{X, \emptyset, G_i \cup \{\sqrt{3}\}\}$. Then (X, τ) is a space but not a topological space. Assume $A = \{\sqrt{3}\}$, then $A_\tau^\vee = \emptyset$. Only open set contained in A is \emptyset and $\emptyset \subset \emptyset = A_\tau^\vee$. So A is a $g\vee_\tau$ -set. Again suppose $B = \{\sqrt{5}\}$, then $B_\tau^\vee = \emptyset$. The only open set contained in B is the set \emptyset and $\emptyset \subset \emptyset = B_\tau^\vee$. So B is a $g\vee_\tau$ -set. Now suppose $C = A \cup B = \{\sqrt{3}, \sqrt{5}\}$. Then $C_\tau^\vee = \emptyset$. Also C is an open set and $C \subset C$ but $C \not\subset C_\tau^\vee$. Therefore $A \cup B$ is not a $g\vee_\tau$ -set.

Similarly, intersection of two $g\wedge_\tau$ -sets is not always a $g\wedge_\tau$ -set as revealed by the under-noted example.

Example 3.33 : Consider the space (X, τ) as in Example 3.32 where $A = \{\sqrt{3}\}$ and $B = \{\sqrt{5}\}$ are $g\vee_\tau$ -sets and $A \cup B$ is not a $g\vee_\tau$ -set. Therefore $X - (A \cup B)$ is not a $g\wedge_\tau$ -set.

Now $X - A$ and $X - B$ are $g\wedge_\tau$ -sets. So $(X - A) \cap (X - B) = X - (A \cup B)$ is not a $g\wedge_\tau$ -set.

Remark 3.34 : Clearly if A is \vee_τ -set, then A is $g\vee_\tau$ -set. But the converse is not necessarily true as shown in the following example.

Example 3.35 : Let $X = R$, G_i 's be the countable subsets of irrationals in R and let $\tau = \{X, \emptyset, G_i\}$. Then (X, τ) is a space but not a topological space. Suppose $A = Q$. Then $(X - A)_\tau^\wedge = X$ and $\overline{X - A} = X$. Therefore $X - A$ is a $g\wedge_\tau$ -set, since the only closed set containing $X - A$ is X . So A is $g\vee_\tau$ -set. Now $X - A = R - Q \neq X = (X - A)_\tau^\wedge$. So $X - A$ is not \wedge_τ -set. Therefore A is not \vee_τ -set.

Theorem 3.36 : For each $x \in X$, $\{x\}$ is either open or a $g\vee_\tau$ -set.
The proof is straightforward and so is omitted.

Note 3.37 : In view of above theorem it follows that for any $x \in X$, either $\{x\}$ is g^* -open or $\{x\}$ is $g\vee_\tau$ -set.

Theorem 3.38 : Let A be a $g\vee_\tau$ -set in a space (X, τ) . If $A_\tau^\vee \cup (X - A) \subset F$, F is closed, then $F = X$.

Proof. Let A be a $g\vee_\tau$ -set in a space (X, τ) . Then $U \subset A_\tau^\vee$ whenever $U \subset A$ and U is open. Let F be a closed set such that $A_\tau^\vee \cup (X - A) \subset F$. This implies $F^c \subset (A_\tau^\vee \cup A^c)^c = (A_\tau^\vee)^c \cap A$ i.e. $X - F \subset (X - A_\tau^\vee) \cap A \dots (1)$. Therefore $X - F \subset A$. Since $X - F$ is open and A is $g\vee_\tau$ -set, then $X - F \subset A_\tau^\vee$. From (1) we have $X - F \subset X - A_\tau^\vee = (A_\tau^\vee)^c$. Therefore $X - F \subset A_\tau^\vee \cap (A_\tau^\vee)^c = \emptyset$ which implies $X = F$. \square

Corollary 3.39 : Let A be a $g\vee_\tau$ -set in a space (X, τ) . Then $A_\tau^\vee \cup (X - A)$ is closed if and only if A is a \vee_τ -set.

Proof. Let A be a $g\vee_\tau$ -set in a space (X, τ) and let $A_\tau^\vee \cup (X - A)$ be closed. Then by Theorem 3.38, $A_\tau^\vee \cup (X - A) = X$ which implies that $(A_\tau^\vee)^c \cap A = \emptyset$. So $A \subset \{(A_\tau^\vee)^c\}^c = A_\tau^\vee$. Again $A_\tau^\vee \subset A$. Therefore $A = A_\tau^\vee$ which implies that A is \vee_τ -set.

Conversely, Let A be a $g\vee_\tau$ -set and also \vee_τ -set. Therefore $A \subset A_\tau^\vee$, since $A = A_\tau^\vee$. So $(A_\tau^\vee)^c \cap A = \emptyset$ which implies that $A_\tau^\vee \cup (X - A) = X$. Therefore $A_\tau^\vee \cup (X - A)$ is closed. \square

Theorem 3.40 : If A is a \vee_τ -set, then A is g^* -open if and only if A is open.

Proof. Let A be open. Then clearly A is g^* -open. Conversely, Let A be a \vee_τ -set and let A be g^* -open. So by Theorem 3.14 there is an open set $U \subset A$ such that $A_\tau^\vee \subset U$. This implies that $A \subset U$, since $A = A_\tau^\vee$. Therefore $A = U$ and so A is open. \square

In particular, since $(A_\tau^\vee)^\vee = A_\tau^\vee$, so A_τ^\vee is a \vee_τ -set. Therefore A_τ^\vee is g^* -open if and only if A_τ^\vee is open.

4. T_ω -Space

Definition 4.1 [5] : A space (X, τ) is said to be T_ω -space if every g^* -closed set is closed.

In Theorem 16 [5], it is shown that every T_ω -space is T_0 -space. But the converse is not true as shown in the Example 6 [5]. Also, in Examples 6 and 7 [5] it has been shown that T_ω and T_1 axioms in a space are independent of each other.

Definition 4.2 [5]) : For any $E \subset X$ let $\overline{E}^* = \cap\{A : E \subset A, A \text{ is } g^*\text{-closed set in } X\}$, then \overline{E}^* is called g^* -closure of E .

We consider the following sets which will be used frequently in the sequel:

$$C = \{A : \overline{(X - A)} \text{ is closed}\} \text{ and } C^* = \{A : \overline{(X - A)}^* \text{ is } g^*\text{-closed}\}.$$

Theorem 4.3 [5]) : A space (X, τ) is T_ω if and only if

- (a) for each $x \in X$, $\{x\}$ is either open or closed and
- (b) $C = C^*$.

Theorem 4.4 : The following are equivalent:

- (1) (X, τ) is T_ω -space.
- (2) Every $g\wedge_\tau$ -set is \wedge_τ -set and $C = C^*$.
- (3) Every $g\vee_\tau$ -set is \vee_τ -set and $C = C^*$.

Proof. (1) \Rightarrow (2): Let (X, τ) be T_ω -space and let A be a $g\wedge_\tau$ -set. We wish to prove $A \supset A_\tau^\wedge$. If not, suppose that $x \in A_\tau^\wedge$ but $x \notin A$. By Theorem 4.3, $\{x\}$ is either open or closed. We discuss two cases:

case (i): Suppose $\{x\}$ is open. So $X - \{x\}$ is a closed set containing A . Since A is $g\wedge_\tau$ -set, we get $A_\tau^\wedge \subset (X - \{x\})$ which implies that $x \notin A_\tau^\wedge$, a contradiction.

Case (ii): Suppose $\{x\}$ is closed. Then $X - \{x\}$ is an open set containing A . But $x \in A_\tau^\wedge = \cap\{U, U \in \tau, U \supset A\} \subset X - \{x\}$, a contradiction. Hence in any case $A \supset A_\tau^\wedge$ and so $A = A_\tau^\wedge$ which implies that A is \wedge_τ -set. Also by Theorem 4.3, $C = C^*$.

(2) \Rightarrow (3): Let A be a $g\vee_\tau$ -set and $C = C^*$. Then by Definition 3.15, $X - A$ is $g\wedge_\tau$ -set. By supposition $X - A$ is a \wedge_τ -set. So A is a \vee_τ -set.

(3) \Rightarrow (2): Let B be a $g\wedge_\tau$ -set and $C = C^*$. So $X - B$ is $g\vee_\tau$ -set. By supposition, $X - B$ is \vee_τ -set. So B is \wedge_τ -set.

(2) \Rightarrow (1) : Let $x \in X$ and $C = C^*$. We will prove $\{x\}$ is either open or closed. If $\{x\}$ is not open, $X - \{x\}$ is not closed. So X is the only closed set containing $X - \{x\}$. Also $(X - \{x\})_\tau^\wedge \subset X$. Therefore by definition, $X - \{x\}$ is a $g\wedge_\tau$ -set. By supposition, $X - \{x\}$ is \wedge_τ -set i.e. $(X - \{x\})_\tau^\wedge = X - \{x\}$. So $X - \{x\}$ must be open. So $\{x\}$ is a closed set. Therefore the space is T_ω . \square

5. λ^* -closed sets and λ^* -open sets in a space

Definition 5.1 : A subset A of a space (X, τ) is said to be λ^* -closed if $A = L \cap \overline{F}$ where L is a \wedge_τ -set and F is a subset of X .

Clearly every \wedge_τ -set is λ^* -closed and closed set is λ^* -closed. But the converse may not be true as shown in the following example.

Example 5.2 : Let $X = R - Q$, $\tau = \{X, \emptyset, G_i\}$, where G_i be the all countable subsets of X . So (X, τ) is a space but not a topological space. Let A be the set of all irrational numbers

in $(0, \infty)$. Then clearly A is g^* -closed but A is not closed. Now $A_\tau^\wedge = X$ and $\overline{A} = A$. Hence $A_\tau^\wedge \cap \overline{A} = A$. So A is λ^* -closed. But $A_\tau^\wedge \neq A$ which implies A is not \wedge_τ -set.

Lemma 5.3 : For a subset A of a space (X, τ) the following are equivalent:

- (i) A is λ^* -closed.
- (ii) $A = A_\tau^\wedge \cap \overline{F}$, $F \subset X$.
- (iii) $A = L \cap \overline{A}$, L is a \wedge_τ -set.
- (iv) $A = A_\tau^\wedge \cap \overline{A}$.

Proof. (i) \Leftrightarrow (ii): Let A be λ^* -closed i.e. $A = L \cap \overline{F}$ where $L = L_\tau^\wedge$, $F \subset X$. Then $A \subset L$ which implies that $A_\tau^\wedge \subset L_\tau^\wedge$ and $A \subset \overline{F}$. Therefore $A \subset A_\tau^\wedge \cap \overline{F} \subset L_\tau^\wedge \cap \overline{F} = L \cap \overline{F} = A$. Hence $A = A_\tau^\wedge \cap \overline{F}$.

Conversely, suppose $A_\tau^\wedge \cap \overline{F} = A$ where $F \subset X$. Since A_τ^\wedge is a \wedge_τ -set then A is λ^* -closed.

(i) \Leftrightarrow (iii): Assume $A = L \cap \overline{F}$ where $L = L_\tau^\wedge$ and $F \subset X$. Then $A \subset L$ and $A \subset \overline{F}$. So $\overline{A} \subset \overline{\overline{F}} = \overline{F}$. Therefore $A \subset L \cap \overline{A} \subset L \cap \overline{F} = A$ and hence $A = L \cap \overline{A}$.

Converse part directly follows from definition.

(i) \Leftrightarrow (iv): Assume $A = L \cap \overline{F}$ where $L = L_\tau^\wedge$ and $F \subset X$. Then $A \subset L$, so $A_\tau^\wedge \subset L_\tau^\wedge$. Again $A \subset \overline{F}$. This implies that $\overline{A} \subset \overline{\overline{F}} = \overline{F}$. Therefore $A \subset A_\tau^\wedge \cap \overline{A} \subset L_\tau^\wedge \cap \overline{F} = L \cap \overline{F} = A$ and hence $A = A_\tau^\wedge \cap \overline{A}$.

Converse part directly follows from definition. \square

Theorem 5.4 : If A is λ^* -closed and $g\wedge_\tau$ -set such that \overline{A} is closed, then A is a \wedge_τ -set.

Proof. Suppose that A is λ^* -closed and $g\wedge_\tau$ -set such that \overline{A} is closed. Since \overline{A} contains A and A is $g\wedge_\tau$ -set, then $A_\tau^\wedge \subset \overline{A}$. Since A is λ^* -closed, $A = A_\tau^\wedge \cap \overline{A}$ which implies $A = A_\tau^\wedge$. Hence A is \wedge_τ -set. \square

Remark 5.5 : From Lemma 5.3 (iv) we can say that a subset A is said to be λ^* closed if A can be expressed as the intersection of all open sets and all closed sets containing it.

In view of above remark 5.5 and by Theorem 18 [5] we have the following theorem 5.6. However we are also giving a separate proof of the Theorem 5.6.

Theorem 5.6 : A space (X, τ) is T_ω if and only if every subset of (X, τ) is λ^* -closed and $C = C^*$.

Proof. Suppose every subset of (X, τ) is λ^* -closed and $C = C^*$ and $x \in X$. We shall show that $\{x\}$ is either open or closed. Suppose $\{x\}$ is not open, then $X - \{x\}$ is not closed. Since $X - \{x\}$ is also a λ^* -closed set then $X - \{x\} = (X - \{x\})_\tau^\wedge \cap \overline{(X - \{x\})}$, (by Lemma 5.3(iv)) $= (X - \{x\})_\tau^\wedge \cap X = (X - \{x\})_\tau^\wedge$. Therefore $X - \{x\}$ is a \wedge_τ -set. So $X - \{x\}$ is an open set which implies $\{x\}$ is closed. Then by Theorem 4.3, (X, τ) is T_ω -space.

Conversely, suppose that (X, τ) is T_ω -space and $A \subset X$. Then by Theorem 4.3, every singleton is either open or closed and $C = C^*$. So each $x \in X - A$, either $\{x\} \in \tau$ or $(X - \{x\}) \in \tau$. Let $A_1 = \{x : x \in X - A, \{x\} \in \tau\}$, $A_2 = \{x : x \in X - A, X - \{x\} \in \tau\}$, $L = \cap[X - \{x\} : x \in A_2] = X - A_2$ and $F = \cap[X - \{x\} : x \in A_1] = X - A_1$.

Note that L is a \wedge_τ -set i.e. $L = L_\tau^\wedge$ and $F = \overline{F}$.

Now, $L \cap \overline{F} = (X - A_2) \cap (X - A_1) = X - (A_1 \cup A_2) = X - (X - A) = A$. Thus A is λ^* -closed. \square

Remark 5.7 : It is already seen that if a subset A is closed then A is g^* -closed and λ^* -closed. But the converse of this result may not hold in a space as seen in Example 5.2, although it is true in a generalised topological space [10]. However, it is true in a space if the additional condition that $C = C^*$ holds which is shown in the following Theorem 5.8.

Theorem 5.8 : If A is g^* -closed and λ^* -closed and satisfies the condition $C = C^*$, then A is a closed set.

Proof. Suppose A is g^* -closed and λ^* -closed satisfying the condition $C = C^*$. Since A is g^* -closed, there is a closed set F containing A such that $F \subset A_\tau^\wedge$. Since $A \subset F$, $\overline{A} \subset \overline{F}$ and so $\overline{A} \subset F \subset A_\tau^\wedge$. Again since A is λ^* -closed, $A = A_\tau^\wedge \cap \overline{A} = \overline{A}$(1).

Now A is g^* -closed, so $\overline{A^*} = A$, therefore $(X - A) \in C^*$. Since $C = C^*$, $(X - A) \in C$ which implies that \overline{A} is closed. Therefore, by (1), A is a closed set. \square

Remark 5.9 : Union of two λ^* -closed sets may not be λ^* -closed set as revealed in the following example.

Example 5.10 : Let $X = R - Q$, $\tau = \{X, \emptyset, \{\sqrt{3}\}, G_i \cup \{\sqrt{3}\}\}$ where G_i be all countable subsets of $X - \{\sqrt{2}\}$. Then (X, τ) is a space but not a topological space. Suppose $A = \{\sqrt{2}\}$, then $A_\tau^\wedge = X$ and $\overline{A} = A$ and so $A = A_\tau^\wedge \cap \overline{A}$. This implies that A is λ^* -closed. Again suppose that $B = \{\sqrt{3}\}$, then B is also λ^* -closed, since $B = B_\tau^\wedge$ and $\overline{B} = X$. Now let $C = \{\sqrt{2}, \sqrt{3}\}$. Then $C_\tau^\wedge = X$ and $\overline{C} = X$. So $C_\tau^\wedge \cap \overline{C} = X \neq C$ which implies that C is not λ^* -closed.

Remark 5.11 : It is shown in [5] that in a bicomact space (Alexandroff space), g^* -closed sets may not be bicomact and likewise, λ^* -closed sets in a bicomact space may not be bicomact as shown in the following example.

Example 5.12 : Let $X = R - Q$, where G_i be the all countable subsets of $X - \{\sqrt{2}\}$, A_i be the all cofinite subsets of X and $\tau = \{X, \emptyset, G_i, A_i\}$. Then (X, τ) is a space but not a topological space. Clearly (X, τ) is a bicomact space. Suppose A is the set of all irrationals in $(0, 1)$. Then there is no open set of G_i type containing A and so $A_\tau^\wedge = \cap\{X - \{\alpha\} : \alpha \in X - A\} = A$. Therefore A is a \wedge_τ -set which implies A is a λ^* -closed set. But A is not bicomact since $\{\{r\} : r \in A\}$ forms an open cover for A which has no finite subcover.

Definition 5.13 : A subset A of X is said to be λ^* -open set if $X - A$ is λ^* -closed set.

Theorem 5.14 : A subset A of X is λ^* -open set if and only if $A = M \cup \text{Int}(V)$ where M is a \vee_τ -set and V is a subset of X .

Proof. Let A be a λ^* -open set. Then $X - A$ is a λ^* -closed set. So $X - A = L \cap \overline{F}$, where L is a \wedge_τ -set and $F \subset X$. Therefore $X - L \subset A$ and $X - \overline{F} \subset A$. So by Theorem 2.8,

$A = L^c \cup (\overline{F})^c = M \cup \text{Int}(X - F) = M \cup \text{Int}(V)$ where $M = X - L$, a \vee_τ -set and $V = X - F \subset X$.

Conversely, Let $A = M \cup \text{Int}(V)$, M is a \vee_τ -set and $V \subset X$. So $A^c = M^c \cap (\text{Int}(V))^c$ where M^c is a \wedge_τ -set and $(\text{Int}(V))^c = \overline{X - V}$ by Theorem 2.8. Therefore $X - A = L \cap \overline{F}$, where $L = M^c$ and $F = X - V$. Therefore $X - A$ is a λ^* -closed set and hence A is a λ^* -open set. \square

Remark 5.15 : Clearly \vee_τ -sets are λ^* -open sets and open sets are λ^* -open sets. On the other hand if a set A is open and $g\vee_\tau$ -set then A is a \vee_τ -set. Again if A is λ^* -open, g^* -open and satisfies the condition $C = C^*$, then A is open.

Theorem 5.16 : A is λ^* -open if and only if $A = A_\tau^\vee \cup \text{Int}(A)$.

Proof. Let A be λ^* -open. Then $A = M \cup \text{Int}(V)$ where M is a \vee_τ -set and $V \subset X$. Since $M \subset A$, $M_\tau^\vee \subset A_\tau^\vee$ and since $\text{Int}(V) \subset A$, $\text{Int}(\text{Int}(V)) \subset \text{Int}(A)$. So, $A = M \cup \text{Int}(V) = M_\tau^\vee \cup \text{Int}(V) \subset A_\tau^\vee \cup \text{Int}(V) = A_\tau^\vee \cup \text{Int}(\text{Int}(V)) \subset A_\tau^\vee \cup \text{Int}(A)$. Again since $A_\tau^\vee \subset A$ and $\text{Int}(A) \subset A$, $A_\tau^\vee \cup \text{Int}(A) \subset A$. Therefore we get $A = A_\tau^\vee \cup \text{Int}(A)$.

Conversely, suppose $A = A_\tau^\vee \cup \text{Int}(A)$. Since A_τ^\vee is a \vee_τ -set and $A \subset X$, A is λ^* -open, by Theorem 5.14. \square

In [10] it is seen that the collection of all λ_μ -open sets forms a generalised topology μ on X , but unlikely the collection of λ^* -open sets does not form a space structure σ on X as shown in the following example.

Example 5.17 : Intersection of two λ^* -open sets may not be a λ^* -open set.

Consider the sets X, A, B, C and τ as in Example 5.10. Then A, B are λ^* -closed sets and so $X - A$ and $X - B$ are λ^* -open sets. But $C = A \cup B$ is not λ^* -closed. Thus $(X - A) \cap (X - B) = X - (A \cup B)$ is not λ^* -open.

6. $T_{\frac{\omega}{4}}$ -space, $T_{\frac{3\omega}{8}}$ -space, $T_{\frac{5\omega}{8}}$ -space

Definition 6.1 : A space (X, τ) is called $T_{\frac{\omega}{4}}$ -space if for every finite subset P of X and for every $y \in X - P$, there exists a set A_y containing P and disjoint from $\{y\}$ such that A_y is either open or closed.

Theorem 6.2 : Every $T_{\frac{\omega}{4}}$ -space is a T_0 -space.

Proof. Suppose that x, y are two distinct points in X . Since the space is $T_{\frac{\omega}{4}}$, then for every $y \in X - \{x\}$ there exists a set A_y such that for every $\{x\} \subset A_y$ and $\{y\} \not\subset A_y$ where A_y is either open or closed. This implies $T_{\frac{\omega}{4}}$ -space is T_0 -space. \square

But the converse may not be true as shown in the Example 6.5.

Theorem 6.3 : A space (X, τ) is $T_{\frac{\omega}{4}}$ if and only if every finite subset of X is λ^* -closed.

Proof. Suppose (X, τ) is $T_{\frac{\omega}{4}}$ -space and P is a finite subset of X . So for every $y \in X - P$ there is a set A_y containing P and disjoint from $\{y\}$ such that A_y is either open or closed. Let L be the intersection of all open sets A_y and F be the intersection of all closed sets A_y .

Then $L = L_\tau^\wedge$ i.e. L is a \wedge_τ -set and $\overline{F} = F$. Therefore $P = L \cap F = L \cap \overline{F}$. So P is λ^* -closed.

Conversely, let P be a finite set. So by the condition it is λ^* -closed. Then by Lemma 5.3(iii) $P = L \cap \overline{P}$, where L is a \wedge_τ -set. Let $y \in X - P$. If $y \notin \overline{P}$ then there exists a closed set $F = A_y$ containing P such that $\{y\} \not\subset A_y$. Again if $y \in \overline{P} - P$, then $y \notin L$ and so $y \notin U$ for some open set $U = A_y$ containing L , since $L_\tau^\wedge = L$. So $P \subset U$. Hence (X, τ) is $T_{\frac{\omega}{4}}$ -space. \square

Theorem 6.4 : A space (X, τ) is T_0 if and only if every singleton of X is λ^* -closed.

Proof. Let the space (X, τ) be T_0 and $x \in X$. Take a point $y \in X - \{x\}$. So there exists a set A_y containing $\{x\}$ but $y \notin A_y$ such that A_y is either open or closed. Let L and F be the intersection of all such open sets A_y and all such closed sets A_y respectively. Then $\overline{F} = F$ and so $\{x\} = L \cap F = L \cap \overline{F}$ where L is a \wedge_τ -set. This implies that $\{x\}$ is a λ^* -closed set by definition.

Conversely, let $x \in X$. So $\{x\}$ is λ^* -closed. Then $\{x\} = \{x\}_\tau^\wedge \cap \overline{\{x\}}$ by Lemma 5.3(iv). Suppose $y \in X - \{x\}$. If $y \notin \overline{\{x\}}$, then there exists a closed set F containing $\{x\}$ but not containing y . If $y \in \overline{\{x\}}$ then $y \notin \{x\}_\tau^\wedge$. Thus $y \notin V$ for some open set V containing $\{x\}$. Hence the space (X, τ) is T_0 . \square

Example 6.5 : Example of a T_0 -space which is not $T_{\frac{\omega}{4}}$.

Consider the space (X, τ) , the subsets A, B, C as in the Example 5.10 where we see that every singleton in X is λ^* -closed. For, let $r \in X, r \neq \sqrt{2}, \sqrt{3}$ then $\{r\}_\tau^\wedge = \{\sqrt{3}, r\}$ and $\overline{\{r\}} = \{\sqrt{2}, r\}$. Therefore $\{r\} = \{r\}_\tau^\wedge \cap \overline{\{r\}}$. So $\{r\}$ is λ^* -closed. Therefore the space (X, τ) is T_0 -space. But it is proved there that the finite subset C is not a λ^* -closed set. So by Theorem 6.3, (X, τ) is not $T_{\frac{\omega}{4}}$ -space.

Definition 6.6 : A space (X, τ) is called $T_{\frac{3\omega}{8}}$ -space if for every countable subset P of X and for every $y \in X - P$, there exists a set A_y containing P and disjoint from $\{y\}$ such that A_y is either open or closed.

Theorem 6.7 : A space (X, τ) is $T_{\frac{3\omega}{8}}$ -space if and only if every countable subset of X is λ^* -closed. Proof is similar to the proof of Theorem 6.3, so is omitted.

Definition 6.8 : A space (X, τ) is called $T_{\frac{5\omega}{8}}$ -space if for any subset P of X and for every $y \in X - P$, there exists a set A_y containing P and disjoint from $\{y\}$ such that A_y is either open or closed.

Theorem 6.9 : A space (X, τ) is $T_{\frac{5\omega}{8}}$ -space if and only if for every subset E of X is λ^* -closed. Proof is similar to the proof of Theorem 6.3, so is omitted.

Note that $T_{\frac{5\omega}{8}}$ axiom does not imply $C = C^*$.

Remark 6.10 : It follows from Theorem 5.6, Theorem 6.9, Theorem 6.7, Theorem 6.3 that every T_ω -space is $T_{\frac{5\omega}{8}}$ -space and $T_{\frac{5\omega}{8}}$ -space is $T_{\frac{3\omega}{8}}$ -space and $T_{\frac{3\omega}{8}}$ -space is $T_{\frac{\omega}{4}}$ -space.

However, the converse of each implication may not be true as shown in the undermentioned examples.

Example 6.11 : Example of a $T_{\frac{5\omega}{8}}$ -space which is not T_ω -space.

Let $X = R - Q$, $\tau = \{X, \emptyset, G_i\}$, where G_i be the all countable subsets of X . Then (X, τ) is a space but not a topological space. We see that the space (X, τ) is $T_{\frac{5\omega}{8}}$ -space, since any finite set, countable set and any uncountably infinite set are also λ^* -closed sets. But the set of all irrationals in $(0, 1)$ is g^* -closed but not closed which implies the space is not T_ω -space.

Example 6.12 : Example of a $T_{\frac{3\omega}{8}}$ -space which is not $T_{\frac{5\omega}{8}}$ -space.

Let $X = R - Q$, G_i be the countable subsets of X containing $\sqrt{2}$ and $\tau = \{X, \emptyset, G_i\}$. So (X, τ) is a space but not a topological space. Take any countable subset $A \subset X$. Then if $\sqrt{2} \in A$, A is an open set. Therefore $A = A_\tau^\wedge$ which implies that A is λ^* -closed. If $\sqrt{2} \notin A$, $A_\tau^\wedge = \{\sqrt{2}\} \cup A$ and $\overline{A} = A$. This implies that A is λ^* -closed. So (X, τ) is a $T_{\frac{3\omega}{8}}$ -space. Now let B be an uncountably infinite subset of X containing the point $\sqrt{2}$. Therefore $B_\tau^\wedge = X$ and $\overline{B} = X$. Therefore $B_\tau^\wedge \cap \overline{B} \neq B$ which implies that B is not λ^* -closed. So (X, τ) is not $T_{\frac{5\omega}{8}}$ -space.

Example 6.13 : Example of a $T_{\frac{\omega}{4}}$ -space which is not $T_{\frac{3\omega}{8}}$ -space.

Suppose that $X = R - Q$ and $X^* = \{2\} \cup X$. Let $\tau = \{\emptyset, X^*, \{2\} \cup (X - A); A \subset X\}$ where A 's are the finite subsets of X . Therefore (X^*, τ) is a topological space so a space also. Take any finite subset $E \subset X^*$, we get the following observations:

(i) if $2 \in E$, $E_\tau^\wedge = \cap\{(X - \{\alpha\}) \cup \{2\}, \alpha \in X - E\} = \{2\} \cup E = E$ which implies that E is a \wedge_τ -set. Therefore E is a λ^* -closed set.

(ii) if $2 \notin E$, E is a closed set which implies that E is λ^* -closed.

So (X^*, τ) is a $T_{\frac{\omega}{4}}$ -space.

Now suppose that Y is a countably infinite subset of X , so $2 \notin Y$. Here closed sets are finite. Therefore $\overline{Y} = X^*$ and $Y_\tau^\wedge = \{2\} \cup Y$. Thus $\overline{Y} \cap Y_\tau^\wedge = \{2\} \cup Y \neq Y$. Therefore Y is not λ^* -closed. Hence (X, τ) is not $T_{\frac{3\omega}{8}}$ -space.

Theorem 6.14 : A space (X, τ) is T_1 if and only if it is T_0 and R_0 .

Proof. Let (X, τ) be a T_1 -space. Obviously then it is T_0 . Let $A \subset X$, A be open, $x \in A$. Since the space is T_1 , for $x, y \in X$ and $x \neq y$ there are open sets U, V such that $x \in U$ and $y \notin U$ and $y \in V$ and $x \notin V$. Hence y cannot be an accumulation point of $\{x\}$. Therefore no point lying outside $\{x\}$ can be an accumulation point of $\{x\}$, so $\{x\}' \subset \{x\}$. Hence $\overline{\{x\}} = \{x\} \subset A$. So (X, τ) is R_0 -space.

Conversely, let (X, τ) be T_0 and R_0 . So for $x, y \in X$ and $x \neq y$, either $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$. Suppose that $x \notin \overline{\{y\}}$. Then there exists a closed set F containing y such that $x \notin F$. Therefore $x \in X - F$, an open set and $y \notin X - F$. Since the space is also R_0 , $\overline{\{x\}} \subset X - F$. So $\overline{\{x\}} \cap F = \emptyset$ which implies that $\overline{\{x\}} \cap \{y\} = \emptyset$. Hence $y \notin \overline{\{x\}}$ and so y is not a limiting point of $\{x\}$. Therefore there exists an open set V containing y such that $x \notin V$. Since $x, y \in X$ and $x \neq y$, we get two open sets $X - F$ and V containing x, y respectively and $y \notin X - F$ and $x \notin V$. Thus (X, τ) is T_1 -space. \square

Theorem 6.15 : A space (X, τ) is T_1 if and only if every singleton is \wedge_τ -set.

Proof. Let (X, τ) be T_1 . So by Theorem 6.14, (X, τ) is T_0 and R_0 . Since (X, τ) is T_0 , every singleton is λ^* -closed, by Theorem 6.4. Suppose $x \in X$, then $\{x\}$ is λ^* -closed. So $\{x\} = \{x\}_\tau^\wedge \cap \overline{\{x\}}$, by Lemma 5.3 (iv). We claim that $\{x\} = \{x\}_\tau^\wedge$. If not, there exists $y \in \{x\}_\tau^\wedge - \{x\}$. So $y \notin \overline{\{x\}}$, and hence there is a closed set F , $F \supset \{x\}$ such that $y \notin F$. Therefore $y \in X - F$, an open set. Again since (X, τ) is R_0 , $\overline{\{y\}} \subset X - F$. Thus $\overline{\{y\}} \cap F = \emptyset$. Since $x \in F$, $x \notin \overline{\{y\}}$. Therefore there exists an open set V containing x but $y \notin V$. This implies that $y \notin \{x\}_\tau^\wedge$, a contradiction. Hence $\{x\}$ is a \wedge_τ -set.

Conversely, let $x, y \in X$ and $x \neq y$. So $y \notin \{x\}$. By supposition $\{x\}$ and $\{y\}$ are \wedge_τ -sets i.e. $\{x\} = \{x\}_\tau^\wedge$ and $y \notin \{x\}_\tau^\wedge$. Therefore there exists an open set V' such that $x \in V'$, but $y \notin V'$. Similarly, since $\{y\} = \{y\}_\tau^\wedge$, there exists an open set U' such that $y \in U'$ and $x \notin U'$. Hence x, y are weakly separated by open sets V' and U' respectively and (X, τ) is T_1 -space. \square

Definition 6.16 : A space (X, τ) is said to be weak R_0 -space if every λ^* -closed singleton is a \wedge_τ -set.

Theorem 6.17 : Every R_0 -space is a weak R_0 -space.

Proof. Suppose $x \in X$ and $\{x\}$ is λ^* -closed, then $\{x\} = \{x\}_\tau^\wedge \cap \overline{\{x\}}$ by Lemma 5.3 (iv). We claim that $\{x\}$ is a \wedge_τ -set. If not, then $\{x\} \neq \{x\}_\tau^\wedge$ and so let $y \in \{x\}_\tau^\wedge - \{x\}$. Then $y \notin \overline{\{x\}}$. So there is a closed set F , $F \supset \{x\}$ such that $y \notin F$. This implies that $y \in X - F$, an open set. Since (X, τ) is R_0 -space, $\overline{\{y\}} \subset X - F$. Therefore $\overline{\{y\}} \cap F = \emptyset$. Since $x \in F$, $x \notin \overline{\{y\}} = \{y\} \cup \{y\}'$ where $\{y\}'$ denotes the set of limit points of $\{y\}$. Therefore there exists an open set $V \supset \{x\}$ such that $y \notin V$, since $x \neq y$ and x is not also the limit point of $\{y\}$. This implies that $y \notin \{x\}_\tau^\wedge$, a contradiction. Hence (X, τ) is weak R_0 -space. \square

But the following example shows that the converse of the Theorem 6.17 may not be true.

Example 6.18 : Let $X = R$ and G_i be the countable subsets of $X - Q - \{\sqrt{2}\}$, $\tau = \{X, \emptyset, G_i\}$. Then (X, τ) is a space but not a topological space. We verify which singletons are λ^* -closed. Take $\sqrt{2} \in X$. Now $\{\sqrt{2}\}_\tau^\wedge = X$ and $\overline{\{\sqrt{2}\}} = \{\sqrt{2}\} \cup Q$ which imply that $\{\sqrt{2}\}$ is not λ^* -closed. Rational singletons are not λ^* -closed. For, suppose that $A = \{\frac{1}{2}\}$. Then $A_\tau^\wedge = X$, $\overline{A} = A \cup Q \cup \{\sqrt{2}\} = \{\sqrt{2}\} \cup Q$. Therefore $A_\tau^\wedge \cap \overline{A} = \{\sqrt{2}\} \cup Q \neq A$. Now for any irrationals r except $\sqrt{2}$, $\{r\}$ is λ^* -closed. So (X, τ) is a weak R_0 -space. Also if $B = \{\sqrt{5}\}$ then $\overline{B} = B \cup Q \cup \{\sqrt{2}\}$ and B is an open set but B does not contain \overline{B} , hence the space is not R_0 .

Lemma 6.19 : If every subset of X is \wedge_τ -set, then (X, τ) is T_1 -space.

Proof. Let every subset of X be \wedge_τ -set. Then every singleton is \wedge_τ -set. So by Theorem 6.15, (X, τ) is T_1 -space. \square

Note: 6.20 : The converse is not true as revealed in the following Example 6.21. But the converse is true by imposing additional conditions as given in the Lemma 6.23. Also note

that the converse part is true in a μ -space [10].

Example 6.21 : Let $X = \mathbb{R} - \mathbb{Q}$, G_i be the countable subsets of X and $\tau = \{X, \emptyset, G_i\}$. Then (X, τ) is a space but not a topological space. The space is T_1 by Theorem 6.15. Let B be the set of all irrationals in $(0, 1)$. Then B is not a \wedge_τ -set as $B^\wedge_\tau = X \neq B$.

Definition 6.22 (cf. Definition 15 [5]) : A space (X, τ) is said to be strongly symmetric if $\{x\}$ is g^* -closed for each $x \in X$.

Lemma 6.23 : If (X, τ) is a strongly symmetric T_1 -space and satisfies the condition $C = C^*$, then every subset of X is a \wedge_τ -set.

Proof. Let (X, τ) be a strongly symmetric T_1 -space satisfying the condition $C = C^*$ and let $A \subset X$, $x \in X$ and $x \notin A$. Then, by definition, $\{x\}$ is g^* -closed. Since (X, τ) is T_1 -space, $\{x\}$ is a \wedge_τ -set by Theorem 6.15 and so a λ^* -closed set. Therefore $\{x\}$ is a closed set by Theorem 5.8. Therefore $X - \{x\}$ is an open set containing A . So $A = \bigcap \{X - \{x\}, x \in X - A\}$ which implies A is a \wedge_τ -set. \square

Remark 6.24: If the space (X, τ) is a strongly symmetric T_1 -space and satisfies the condition $C = C^*$, then union and intersection of two λ^* -closed sets are λ^* -closed sets.

Theorem 6.25 [10] : For a space (X, τ) , the following statements are equivalent:

- (1) (X, τ) is T_1
- (2) (X, τ) is T_0 and R_0
- (3) (X, τ) is T_0 and weak R_0 .

Proof. (1) \Rightarrow (2): It follows from Theorem 6.14.

(2) \Rightarrow (3): It follows from Theorem 6.17.

For (3) \Rightarrow (1): let (X, τ) be T_0 and weak R_0 and $\{x\} \subset X$. So by Theorem 6.4, $\{x\}$ is λ^* -closed. Again (X, τ) is weak R_0 , $\{x\}$ is \wedge_τ -set. By Lemma 6.19, (X, τ) is T_1 . \square

Theorem 6.26 : If (X, τ) is a strongly symmetric T_1 -space and satisfies the condition $C = C^*$, then it is T_ω -space.

Proof. Let (X, τ) be strongly symmetric T_1 -space satisfying the condition $C = C^*$ and let A be a g^* -closed set. Then $\overline{A^*} = A$ and so $\overline{A^*}$ is g^* -closed. Therefore $A^c \in C^* = C$ which implies that \overline{A} is closed. Now let $x \in \overline{A} - A$. Since (X, τ) is T_1 and strongly symmetric space, $\{x\}$ is g^* -closed and λ^* -closed, and since $C = C^*$, $\{x\}$ is a closed set by Theorem 5.8. But by Theorem 3.5, $\{x\} \not\subset \overline{A} - A$. Therefore $x \in A$ and so $\overline{A} = A$ which implies that A is a closed set and hence the space (X, τ) is T_ω . \square

Theorem 6.27 : If the space (X, τ) is strongly symmetric, weak R_0 , and satisfies the condition $C = C^*$, then the following are equivalent:

- (1) (X, τ) is T_0
- (2) (X, τ) is T_1
- (3) (X, τ) is T_ω
- (4) (X, τ) is $T_{\frac{5\omega}{8}}$
- (5) (X, τ) is $T_{\frac{3\omega}{8}}$

(6) (X, τ) is $T_{\frac{\omega}{4}}$.

Proof. (1) \Rightarrow (2): It follows from Theorem 6.25.

(2) \Rightarrow (3): It follows from Theorem 6.26.

(3) \Rightarrow (4): It follows from Theorem 5.6 and 6.9.

(4) \Rightarrow (5): It follows from Theorem 6.9 and 6.7.

(5) \Rightarrow (6): It follows from Theorem 6.7 and 6.3.

(6) \Rightarrow (1): It follows from Theorem 6.3 and 6.2. □

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